Modulational Interaction of Secondary Structures (Part 2)

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Abstract

These notes continue from a discussion of disparate scale interaction. In Part 1, the basic picture of disparate scale interaction was presented, and two examples were discussed: Langmuir turbulence and drift wave-zonal flow turbulence. In both cases, the small scale species generates the larger scale structures through turbulent processes (plasma waves exert radiation pressure on ion acoustic waves, drift waves exert Reynolds stress on zonal flows), while the larger scale structures refract the smaller scales through density perturbations. Ray tracing and quasilinear theory were used to describe Langmuir turbulence and drift wave-zonal flow turbulence, and the transfer of energy and momentum between the two respective species. For higher Kubo number, the Zakharov equations were derived to describe the envelope of the smaller scale's field fluctuations. Finally, under an adiabatic approximation, the Zakaharov equations were reduced to a nonlinear Schrodinger equation (NLS), which was used to describe Langmuir collapse.

In this section of the notes, the nonlinear Schrodinger equation is revisited. The NLS is derived starting from the equations for four-wave coupling, under the conditions of isotropic dispersion and separation of envelope and carrier wave scales, showing that the NLS is generic to modal interactions. The topic of Langmuir collapse is revisted and comments are made on the connection to 3D fluid turbulence. Last, the stability of a stationary state of the NLS is studied, and the Lighthill criterion for modulational instability is derived. The modulational stability of the NLS depends on the competition between wave dispersion and a self-focusing potential, which is dependent both on the shape of the linear wave dispersion relation and nonlinear coupling.

1 Another look at the Nonlinear Schrodinger Equation (NLS)

The NLS was previously derived from an adiabatic approximation of the Zakharov equations, in the context of Langmuir turbulence. In this section, the NLS will be shown to be generic to modal interactions, and will be derived from a generic Fourier mode coupling equation with separation between carrier waves and an envelope scale, in the formalism of Falkovich. One possible consequence of modal interactions, in the context of the NLS, has been seen to be Langmuir collapse: the self-focusing of plasma waves due to refraction by self-produced ion acoustic waves. This topic will be revisited with an alternative approach, i.e. by studying the modulational stability of the nonlinear state.

We start from a generic 4-wave coupling description for the Fourier mode amplitude of a carrier wave $a_{\mathbf{k}}$,

$$\frac{\partial}{\partial t}a_{\mathbf{k}} + i\omega_{\mathbf{k}}a_{\mathbf{k}} = -i\int T_{k_1,k_2,k_3}\, a_1^* a_2 a_3\, \delta(\mathbf{k}+\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\, d\mathbf{k_1}d\mathbf{k_2}d\mathbf{k_3},$$

given a wavepacket centered around a carrier scale $\mathbf{k_0}$ and dispersion relation $\omega_{\mathbf{k}} = \omega(\mathbf{k})$.

Since the carrier waves are all of the same scale, the 4-wave matching condition introduces a large scale, via which the carrier waves interact. This represents the envelope scale, $\mathbf{q}$, such that,

$$\mathbf{k} \sim \mathbf{k_0} + \mathbf{q}, \quad q \ll k_0.$$

The frequency of the carrier waves is approximated by

$$\omega(\mathbf{k_0} + \mathbf{q}) \approx \omega_0 + \mathbf{q} \cdot \frac{\partial \omega}{\partial \mathbf{k}} + \frac{1}{2}\, q_i q_j \, \frac{\partial^2 \omega}{\partial k_i \, \partial k_j}. \tag{1}$$

Given that the dispersion relation for plasma waves is isotropic, $\omega(\mathbf{k}) = \omega(|k|)$,

$$\frac{\partial}{\partial k_i} \rightarrow \frac{k_i}{|k|} \frac{\partial}{\partial k},$$

and the second term on the RHS of Eq. (1), is given by:

$$\begin{aligned} q_i q_j \, \frac{\partial^2 \omega}{\partial k_i \, \partial k_j} &= q_i q_j \left[\frac{k_i \, k_j}{k^2} \frac{\partial^2 \omega}{\partial k^2} + \left(\delta_{i,j} - \frac{k_i \, k_j}{k^2} \right) \frac{v_{gr}}{k} \right] \\ &= q_{\|}^2 \omega'' + q_{\perp}^2 \frac{v_{gr}}{k}, \end{aligned}$$

where $q_{\|} = \mathbf{q} \cdot \mathbf{k}/k$, and $v_{gr} = \partial \omega / \partial k$.

Introducing a slowly varying envelope to the wave amplitude,

$$a_{\mathbf{k}}(t) = a_{\mathbf{k}} \, e^{-i\omega_{\mathbf{k}} t} = \psi_{\mathbf{q}}(t) \, e^{-i\omega_0 t},$$

and extracting out the (fast) response of the carrier waves, the (slow) response of the envelope is given by:

$$\left[i\frac{\partial}{\partial t} - \mathbf{q} \cdot \mathbf{v_{gr}} - \frac{q_{\|}^2 \, \omega''}{2} - \frac{q_{\perp}^2 \, v_{gr}}{2k} \right] \psi_q = \int T_{1,2,3} \, \psi^*_{\mathbf{q_1}} \psi_{\mathbf{q_2}} \psi_{\mathbf{q_3}} \, \delta(\mathbf{q} + \mathbf{q_1} - \mathbf{q_2} - \mathbf{q_3}) \, d\mathbf{q_1} d\mathbf{q_2} d\mathbf{q_3} \tag{2}$$

Moving to real space, $\psi(\mathbf{r}) = \int dq \, \psi_{\mathbf{q}} \, e^{i\mathbf{q}\cdot\mathbf{r}}$, the RHS of Eq. (2) becomes,

$$\begin{aligned} \text{FT(RHS)} = &\int d\mathbf{r_1} \, d\mathbf{r_2} \, d\mathbf{r_3} \, \psi^*(r_1) \, \psi(r_2) \, \psi(r_3) \\ &\times \int d\mathbf{q} \, d\mathbf{q_1} \, d\mathbf{q_2} \, d\mathbf{q_3} \, T \, \delta(\mathbf{q} + \mathbf{q_1} - \mathbf{q_2} - \mathbf{q_3}) \, e^{i(\mathbf{q}\cdot\mathbf{r} + \mathbf{q_1}\cdot\mathbf{r_1} - \mathbf{q_2}\cdot\mathbf{r_2} - \mathbf{q_3}\cdot\mathbf{r_3})} \\ = &\, T \int d\mathbf{r_1} \, d\mathbf{r_2} \, d\mathbf{r_3} \, \psi^*(r_1) \, \psi(r_2) \, \psi(r_3) \, \delta(\mathbf{r} - \mathbf{r_1}) \, \delta(\mathbf{r} - \mathbf{r_2}) \, \delta(\mathbf{r} - \mathbf{r_3}) \\ = &\, T \left[\psi^*(\mathbf{r}) \, \psi(\mathbf{r}) \right] \, \psi(\mathbf{r}) \end{aligned}$$

Therefore, the envelope evolves in real space according to,

$$\frac{\partial}{\partial t}\psi(\mathbf{r}) + v_{gr} \nabla_{\|} \psi - i\frac{\omega''}{2} \nabla_{\|}^2 \psi - i\frac{v_{gr}}{2k} \, \nabla_{\perp}^2 \psi = -iT \, |\psi|^2 \, \psi.$$

Eliminating the advective term by moving to a reference frame propagating at the group velocity, $\psi(x,t) \rightarrow \psi(x - v_{gr}t, t)$, and rescaling the perpendicular coordinate, $x_{\perp} \rightarrow (k_0 \omega''/v_{gr})^2 \, x_{\perp}$, we now arrive at the nonlinear Schrodinger equation,

$$i\frac{\partial \psi}{\partial t} + \frac{\omega''}{2} \, \nabla^2 \psi - T \left| \psi^2 \right| \, \psi = 0.$$

In summary, the NLS is recovered as a description of generic 4-wave mode coupling, with the assumptions of isotropic dispersion, a finite spectrum of carrier waves, and separation of wave and envelope scales. This is exactly the form of the NLS derived from the Zakharov equations for Langmuir turbulence, as well the time-dependent Gross-Pitaevski equation for the wavefunction of a Bose-Einstein condensate. Solutions of the NLS, and more significantly stability of solutions, are clearly sensitive to the sign of ω'' and T. This highlights the significance of the structure of the carrier wave dispersion relation, as was shown previously for Langmuir collapse.

Closing comments for Langmuir collapse will be presented in the next section, and modulational stability of the NLS will be presented in the following section.

2 Closing comments on Langmuir collapse

2.1 Summary of demonstration of Langmuir collapse

Langmuir turbulence is a nonlinear state of coexistence of high-frequency plasma waves and self-generated large-scale ion-acoustic waves. The ion-acoustic waves refract the plasma waves by causing large-scale density fluctuations, while the plasma waves exert a ponderomotive radiation pressure to drive the ion-acoustic waves. The Zakharov equations describe these two nonlinear effects in the envelope of plasma wave field fluctuations $\xi(x,t)$ and the acoustic wave density fluctuations $\tilde{\rho}(x,t)$.

$$2i\omega_{p_0}\frac{\partial\xi}{\partial t} = -\alpha v_{th}^2\,\nabla^2\xi + \frac{\tilde{\rho}}{\rho_0}\omega_{p_0}\xi$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\tilde{\rho}}{\rho_0}\right) = c_s^2\,\nabla^2\left(\frac{\tilde{\rho}}{\rho_0} + \frac{|\xi|^2}{8\pi\rho_0 c_s^2}\right)$$

Under the approximation that acoustic wave propagation is the fastest timescale of the system, the density perturbations are assumed to be rapidly set by the radiation pressure,

$$\frac{\tilde{\rho}}{\rho_0} \sim \frac{|\xi|^2}{8\pi\rho_0 c_s^2},$$

and we are left with the adiabatic Zakharov equation,

$$2i\omega_{p_0}\frac{\partial\xi}{\partial t} = -\alpha v_{th}^2\,\nabla^2\xi + \frac{\omega_{p_0}^2}{8\pi\rho_0 c_s^2}\,|\xi|^2\,\xi,$$

which takes exactly the form of the nonlinear Schrodinger equation,

$$i\frac{\partial\xi}{\partial t} + \nabla^2\xi - |\xi|^2\,\xi = 0$$

The adiabatic Zakharov equation has some interesting features. First, the potential energy function is purely real. There is no local dissipation, or natural source of saturation in this problem. Furthermore, the potential is negative (attractive), and depends on the amplitude of fluctuations of the field envelope.

An exact solution for the NLS exists in one dimension, in the form of a soliton. The more interesting case is in three dimensions with spherical symmetry. In this case, the NLS takes the following form,

$$i\frac{\partial\xi}{\partial t} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2\xi) - |\xi|^2\,\xi = 0$$

This system has two conserved quantities;

1. The intensity field, which corresponds to the plasmon number:

$$I_1 = \int dr\, r^2\ |\xi|^2$$

2. The Hamiltonian:

$$I_2 = \int dr\, r^2 \left[\left| \frac{\partial \xi}{\partial r} \right|^2 - \frac{1}{4}(|\xi|^2)^2 \right]$$

The first term in the integrand describes the natural diffraction of plasma waves, while the second term describes the (ponderomotive) self-focusing nature of the plasma wave envelope. It is interesting to note that the total energy can go negative for large amplitude waves.

Langmuir collapse is demonstrated by following the evolution of the mean square radius of the field envelope,

$$\langle r^2 \rangle = \frac{\int dr\, r^4\, |\xi|^2}{\int dr\, r^2\, |\xi|^2} = \frac{\int dr\, r^4\, |\xi|^2}{I_1}.$$

It can be shown from the equations of motion, that,

$$\frac{\partial^2}{\partial t^2} \langle r^2 \rangle < \frac{8 I_2}{I_1},$$

and it follows that,

$$\langle r^2(t) \rangle < \frac{8 I_2}{I_1} t^2 + \frac{\partial}{\partial t} \langle r^2 \rangle \bigg|_{t=0} + \langle r^2 \rangle \bigg|_{t=0}$$

Therefore, if $I_2(t=0) < 0$, i.e. the plasma waves are large enough at initial time, the system collapses to $r = 0$ in finite time, i.e. exhibits a finite time singularity. The time for onset of collapse can be estimated as,

$$t \sim \sqrt{\frac{\langle r^2(t=0) \rangle}{I_2 / I_1}} \sim \frac{l}{\tilde{v}_{pw}},$$

where (after reversing normalization of parameters), l represents the initial size of the plasma wavepacket, and $\tilde{v}$ represents the oscillation velocity of the plasma waves.

2.2 Comments

The stability of the Langmuir turbulence state against collapse boils down to a competition between the diffraction, or the natural spreading of the wave envelope, against the self-focusing ponderomotive potential caused by the wave radiation pressure.

The endgame of the Langmuir collapse is a highly singular structure, which is a hallmark of turbulence. In analogy with the Richardson cascade of 3D fluid turbulence, the Langmuir collapse phenomenon describes transport of wave energy to small scales. The limit of the collapse can be approximated by k_0, the wavelength of the original carrier plasma wave. The validity of the envelope approximation ($E \sim \xi(x,t) e^{-i\omega_0 t + i k_0 x}$) is expected to break down at this timescale. In further analogy with 3D fluid turbulence, k_0 can be described as a dissipation scale, the final small scale that fluctuation energy transfers down to. However, it is important to note that the Richardson cascade was formulated as a stationary, local, self-similar process carrying energy from stirring scales to singular shear layer at small scales, while the Langmuir collapse is a dynamical concentration of wave energy to small scale.

The problem of Langmuir turbulence and Langmuir collapse describes a self organization mechanism in the absence of saturation, and acts as a paradigm for the origin of nonlinear structures and nonlinear evolution. A similar type of analysis can be carried out for the drift wave-zonal flow system, for the existence of a singular zonal flow shear layer. The drift wave-zonal flow turbulence problem is complicated by the nonisotropic dispersion relation and the fact that the drift wave is a backward wave, i.e. $v_{gr}/v_{ph} < 0$. These issues are discussed for the case of ETG turbulence in Gurcan and Diamond (2003).

3 Modulational stability of nonlinear Schrodinger equation

In this section, we examine solutions of the nonlinear Schrodinger equation, and we are particularly interested in when and how stationary solutions become unstable to modulations. This more rigorous discussion will enlighten the discussion of Langmuir collapse described previously.

Starting from the NLS,

$$i\frac{\partial \psi}{\partial t} + \frac{\omega''}{2}\nabla^2\psi - T\left|\psi^2\right|\psi = 0,$$

we look for solutions of the type $\psi \sim Ae^{i\phi}$, and we are interested in the stability of perturbations in the amplitude $\tilde{A}$ and phase $\tilde{\phi}$.

We note that the stationary solution is the condensate,

$$\psi = \psi_0 e^{-iA_0^2 Tt},$$

which is a spatially uniform, oscillating solution. The oscillation rate is set by the nonlinearity, and there is no effect of diffraction.

We are now interested in the stability of modulations around the stationary state, i.e. we would like to answer the question of whether these modulations will grow or decay. Taking $\psi \sim (A+\tilde{A})\,e^{i(\phi+\tilde{\phi})}$, linearizing in $\tilde{A}$ and $\tilde{\phi}$, and looking at 1D modulations in the ξ direction,

$$i\left[\frac{\partial \tilde{A}}{\partial t}\,e^{i\phi} + i\frac{\partial \tilde{\phi}}{\partial t}Ae^{i\phi}\right] = \omega''\left(\frac{\partial^2\tilde{A}}{\partial\xi^2} + iA\frac{\partial^2\tilde{\phi}}{\partial\xi^2}\right)\,e^{i\phi} - T\left|A\right|^2\,\tilde{A}\,e^{i\phi} \tag{3}$$

Defining $\tilde{K} = \partial\tilde{\phi}/\partial\xi$, the real part of Eq.(3) describes the time evolution of the gradient of the phase,

$$\frac{\partial \tilde{K}}{\partial t} = -TA\frac{\partial \tilde{A}}{\partial \xi} + \frac{\omega''}{A}\frac{\partial^3\tilde{A}}{\partial\xi^3},$$

and the imaginary part of Eq.(3) describes the time evolution of the amplitude,

$$\frac{\partial \tilde{A}}{\partial t} + \omega'' A\frac{\partial K}{\partial \xi} = 0$$

Finally, describing the amplitude and gradient of the phase by time and spatial envelope scales, as was done previously, $A, K \sim e^{i(q\xi - \Omega t)}$, we obtain the modulation dispersion relation,

$$\Omega^2 = q^2\left(\omega''^2 q^2 + T\omega'' A^2\right)$$

The first term on the RHS is positive and describes diffraction. The second term depends on the nonlinearity (transition rate TA^2), and on the shape of the carrier wave dispersion relation $\partial^2\omega/\partial k^2$, in analogy to the self-focusing effect in the case of Langmuir collapse. Clearly, in order for the system to be unstable to modulations, the carrier wave amplitude must be sufficiently large, and $T\omega''$ must be negative. This is called the Lighthill criterion for modulation instability in NLS.

In passing, we note that the Lighthill criterion requires a combination of linear wave propagation (via ω''), and wave-wave interaction (via T). In analogy with Langmuir collapse, it can be shown that the sign of the Hamiltonian is positive definite for $\omega'' T > 0$ (system is stable to modulations), while the sign of the Hamiltonian is indefinite for $\omega'' T < 0$, and can go negative. Collapse can be recovered in this case.

4 References

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